PRINCIPLES OF ANALYSIS
LECTURE 13 - OPEN AND CLOSED SETS

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1. Neighborhoods

Let $x_0 \in \mathbb{R}$. An *ϵ-neighborhood* of x_0 is an open interval of the form $(x_0 - \epsilon, x_0 + \epsilon)$, where $\epsilon > 0$.

More generally, a *neighborhood* of x_0 is a subset $Q \subset \mathbb{R}$ such that there exists $\epsilon > 0$ with $(x_0 - \epsilon, x_0 + \epsilon) \subset Q$.

A subset $U \subset \mathbb{R}$ is called *open* if

$$\forall u \in U \exists \epsilon > 0 \ni |x - u| < \epsilon \Rightarrow x \in U.$$

Or, in other words, U is open if every point in U is surrounded by an ϵ-neighborhood which is completely contained in U.

Proposition 1. *Let $\mathcal{T}$ denote the collection of all open subsets of $\mathbb{R}$. Then*

- **(a)** $\varnothing \in \mathcal{T}$ *and* $\mathbb{R} \in \mathcal{T}$;
- **(b)** *if* $\mathcal{O} \subset \mathcal{T}$, *then* $\cup\mathcal{O} \in \mathcal{T}$;
- **(c)** *if* $\mathcal{O} \subset \mathcal{T}$ *is finite, then* $\cap\mathcal{O} \in \mathcal{T}$.

Proof.

(a) The condition for openness is vacuously satisfied by the empty set. For $\mathbb{R}$, consider $x \in \mathbb{R}$. Then $(x - 1, x + 1) \subset \mathbb{R}$. Thus $\mathbb{R}$ is open.

(b) Let $\mathcal{O} \subset \mathcal{T}$; that is, $\mathcal{O}$ is a collection of open sets. Select $x \in \cup\mathcal{O}$. Then $x \in U$ for some $U \in \mathcal{O}$. Since U is open, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Since $U \subset \cup\mathcal{O}$, it follows that $(x - \epsilon, x + \epsilon) \subset \cup\mathcal{O}$. Thus $\cup\mathcal{O}$ is open.

(c) Let $\mathcal{O} \subset \mathcal{T}$ be a finite collection of open sets. Since $\mathcal{O}$ is finite, we may write $\mathcal{O} = \{U_1, U_2, \dots, U_n\}$, where U_i is an open set for $i = 1, \dots, n$. If $\cap\mathcal{O}$ is empty, we are done, so assume that it it nonempty, and select $x \in \cap\mathcal{O}$. For each i, there exists ϵ_i such that $(x - \epsilon_i, x + \epsilon_i) \subset U_i$. Set $\epsilon = \min\{\epsilon_1, \dots, \epsilon_n\}$. Then $(x - \epsilon, x + \epsilon) \subset \cap\mathcal{O}$. Thus $\cap\mathcal{O}$ is open. □

Date: October 2, 2003.

Lemma 1. *Let $\mathcal{O}$ be a collection of open intervals. If $\cap\mathcal{O}$ is nonempty, then $\cup\mathcal{O}$ is an open interval.*

Proof. By hypothesis, there exists $x \in \cap\mathcal{O}$. Write $\mathcal{O}$ as a family of sets:

$$\mathcal{O} = \{O_\alpha \mid \alpha \in A\},$$

where A is an indexing set. Now O_α is an open interval; we label its endpoints by letting $O_\alpha = (a_\alpha, b_\alpha)$, where $a_\alpha, b_\alpha \in \overline{\mathbb{R}}$. Set

$$a = \inf\{a_\alpha \mid \alpha \in A\} \quad \text{and} \quad b = \sup\{b_\alpha \mid \alpha \in A\}.$$

Claim: $\cup\mathcal{O} = (a, b)$. We prove both directions of containment.

($\subset$) Let $y \in \cup\mathcal{O}$. Then $y \in O_\alpha$ for some α. Thus $a \leq a_\alpha < y < b_\alpha \leq b$, so $y \in (a, b)$.

($\supset$) Let $y \in (a, b)$. Assume that $y \leq x$; the proof for $y \geq x$ is analogous. Now $a < y$, and since $a = \inf\{a_\alpha \mid \alpha \in A\}$, so there exists $\alpha \in A$ such that $a \leq a_\alpha < y$. Also $x \in O_\alpha$ so $a_\alpha < y \leq x < b_\alpha$; thus $y \in (a_\alpha, b_\alpha) = O_\alpha$, and $y \in \cup\mathcal{O}$. □

Proposition 2. *Let $U \subset \mathbb{R}$. Then U is open if and only if there exists a countable collection $\mathcal{O}$ of disjoint open intervals such that $U = \cup\mathcal{O}$.*

Proof. Put a relation on U by defining $u_1 \sim u_2$ if there exists an open interval O such that $u_1, u_2 \in O$ and $O \subset U$.

Claim 1: This is an equivalence relation. We wish to show that $\sim$ is reflexive, symmetric, and transitive.

Reflexive Let $u \in U$. Since U is open, there exists $\epsilon > 0$ such that $(u - \epsilon, u + \epsilon) \subset U$. Let $O = (u - \epsilon, u + \epsilon)$; then $u \in O$ and $O \subset U$, so $u \sim u$.

Symmetric Let $u_1, u_2 \in U$, and assume $u_1 \sim u_2$. Then there exists an open interval O such that $u_1, u_2 \in O$ and $O \subset U$. But then $u_2, u_1 \in O$, so $u_2 \sim u_1$.

Transitive Let $u_1, u_2, u_3 \in U$, and assume that $u_1 \sim u_2$ and $u_2 \sim u_3$. Then there exist open intervals $O_1, O_2 \subset U$ such that $u_1, u_2 \in O_1$ and $u_2, u_2 \in O_2$. Now $u_2 \in O_1 \cap O_2$, so by the Lemma, $O_1 \cup O_2$ is an interval contained in U and containing u_1 and u_3. Thus $u_1 \sim u_3$.

Claim 2: The equivalence classes of this equivalence relation are open intervals. Let $u \in U$ and let $\overline{u}$ denote the equivalence class of u. For every $v \in \overline{U}$ there exists and open interval O_v such that $u, v \in O_v$ and $O_v \subset U$. Let $\mathcal{O} = \{O_v \mid v \in \overline{u}\}$. Then $u \in \cap\mathcal{O}$, so $\cup\mathcal{O}$ is an open interval; it suffices to show that $\overline{u} = \cup\mathcal{O}$. Clearly $\overline{u} \subset \cup\mathcal{O}$. Moreover, if $x \in \cup\mathcal{O}$, then $x \in O_v$ for some v, so $x \sim u$ and $x \in \overline{u}$. Thus $\overline{u} = \cup\mathcal{O}$.

Claim 3: Distinct equivalence classes have empty intersection. This is true for every equivalence relation.

Claim 4: There are only countably many equivalence classes. Let $\mathcal{O} = \{O_\alpha \mid \alpha \in A\}$ be the collection of equivalence classes, where A is some indexing set. Let $O_\alpha = (a_\alpha, b_\alpha)$. We have seen that there exists $q_\alpha \in \mathbb{Q}$ such that such that $a_\alpha < q_\alpha < b_\alpha$. Let $Q = \{q_\alpha \mid \alpha \in A\}$. Then $|\mathcal{O}| = |A| = |Q| \leq |\mathbb{Q}|$; since $\mathbb{Q}$ is countable, so is $\mathcal{O}$. □

2. Closed Sets

A subset $F \subset \mathbb{R}$ is *closed* if its complement $\mathbb{R} \setminus F$ is open.

Proposition 3. *Let $\mathcal{F}$ denote the collection of all closed subsets of $\mathbb{R}$.*

(a) $\varnothing \in \mathcal{F}$ *and* $\mathbb{R} \in \mathcal{F}$*;*
(b) *if* $\mathcal{C} \subset \mathcal{F}$*, then* $\cap\mathcal{C} \in \mathcal{F}$*;*
(c) *if* $\mathcal{C} \subset \mathcal{F}$ *is finite, then* $\cup\mathcal{C} \in \mathcal{T}$*.*

Proof. Apply DeMorgan's Laws to Proposition 1. □

Proposition 4. *Let $F \subset \mathbb{R}$. Then F is closed if and only if every sequence in F which converges in $\mathbb{R}$ has a limit in F.*

Proof. We prove both directions.

($\Rightarrow$) Suppose that F is closed, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence in F which converges to $a \in \mathbb{R}$. We wish to show that $a \in F$. Suppose not; then $a \in \mathbb{R} \setminus F$. This set is open, so there exists $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subset \mathbb{R} \setminus F$. Thus there exists $N \in \mathbb{Z}^+$ such that $a_n \in \mathbb{R} \setminus F$ for all $n \geq N$. This contradicts that the sequence is in F.

($\Leftarrow$) Suppose that F is not closed; we wish to construct a sequence in F which converges to a point not in F. Since F is not closed, then $\mathbb{R} \setminus F$ is not open. This means that there exists a point $x \in \mathbb{R} \setminus F$ such that for every $\epsilon > 0$, $(x - \epsilon, x + \epsilon)$ is not a subset of $\mathbb{R} \setminus F$; that is, $(x - \epsilon, x + \epsilon)$ contains a point in F. For $n \in \mathbb{Z}^+$, let $x_n \in (x - \frac{1}{n}, x + \frac{1}{n}) \cap F$. Then $\{x_n\}_{n=1}^{\infty}$ is a sequence in F, but $\lim_{n \to \infty} x_n = x \notin F$. □

3. Accumulation Points

A *deleted neighborhood* of x_0 is a set of the form $Q \smallsetminus \{x_0\}$, where Q is a neighborhood of x_0.

Let $S \subset \mathbb{R}$. An *accumulation point* of S is a point $s \in \mathbb{R}$ such that every deleted neighborhood of s contains an element of S.

We note that an accumulation point of a set S may or may not be an element of S.

Proposition 5. *Let $F \subset \mathbb{R}$. Then F is closed if and only if F contains all of its accumulation points.*

Proof. Prove both directions.

($\Rightarrow$) Suppose F is closed, and let $x \in \mathbb{R}$. Suppose $x \notin F$; we show that x is not an accumulation point of F. Since $x \in F$, then $x \in \mathbb{R} \smallsetminus F$, which is open. Therefore there exists $\epsilon > 0$ such that $U = (x - \epsilon, x + \epsilon) \subset \mathbb{R} \smallsetminus F$. Then $U \smallsetminus \{x\}$ is a deleted neighborhood of x whose intersection with F is empty, and x is not an accumulation point of F.

($\Leftarrow$) Suppose F contains all of its accumulation points. We show that the complement of F is open. Let $x \in \mathbb{R} \smallsetminus F$. Then x is not an accumulation point of F. Then there exists a deleted neighborhood U of x such that $U \subset \mathbb{R} \smallsetminus F$. This neighborhood contains a deleted epsilon neighborhood, say $(x - \epsilon, x + \epsilon) \smallsetminus \{x\}$. This set is in the complement of F, and since $x \notin F$, we have $(x - \epsilon, x + \epsilon) \subset \mathbb{R} \smallsetminus F$. Thus $\mathbb{R} \smallsetminus F$ is open, so F is closed. □

Theorem 1 (Bolzano-Weierstrauss Theorem)**.** *Every bounded infinite set of real numbers has an accumulation point.*

Proof. Let S be a bounded infinite set. Since S is infinite, there exists an injective function $s : \mathbb{Z}^+ \to S$; view this as a sequence $\{s_n\}_{n=1}^{\infty}$. This sequence has a monotonic subsequence, say $\{s_{n_k}\}$, which is also bounded and hence convergent, say to $s \in \mathbb{R}$. Suppose that $s = s_{n_K}$ for some K; then, since s is the limit and the sequence is monotonic, it is easy to see that $s_{n_k} = s$ for every $k \geq K$. This contradicts that the sequence was injective. Thus $s \neq s_{n_k}$ for every $k \in \mathbb{Z}^+$.

Now for every $\epsilon > 0$, there exists $K \in \mathbb{Z}^+$ such that $|s_{n_K} - s| < \epsilon$; that is, $s_{n_K} \in (s - \epsilon, s + \epsilon)$, and $s_{n_K} \neq s$. Thus s is an accumulation point for S. □

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